BRAIDING FOR THE QUANTUM gl_2 AT ROOTS OF UNITY

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Abstract. In the preceding papers [KR1, KR2] we started considering the categories of tangles with flat G-connections in their complements, where G is a simple complex algebraic group. The braiding (or the commutativity constraint) in such categories satisfies the holonomy Yang–Baxter equation and it is this property which is essential for our construction of invariants of tangles with flat G-connections in their complements. In this paper, to any pair of irreducible modules over the quantized universal enveloping algebra of gl_2 at a root of unity, we associate a solution of the holonomy Yang–Baxter equation.

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Introduction

The representation theory of quantum groups at roots of unity which we will use was developed in [DCK] and [DCKP]. These quantum groups are finite dimensional over the central Hopf subalgebra generated by ℓ-th powers of generators and root elements. This central Hopf subalgebra is usually denoted by Z_0.

Quansitriangular Hopf algebras and, more generally, tensor categories can be used for construction of invariants of links and 3-manifolds (see [T1] and references therein). In a similar way tensor categories fibered over a braided group can be used for construction of invariants of tangles with flat connections in their complements [T2, KR2]. Less functorially one can say that such invariants can be constructed from any system of solutions to the holonomy Yang–Baxter equation [KR1].

Let $\mathcal{U}_\varepsilon$ be the quantized universal enveloping algebra of gl_2 at a root of unity ε of degree ℓ. The first step towards understanding the braiding structure for quantized universal enveloping algebras was done in [R, Ga]. In this paper we will show that every pair of irreducible modules over $\mathcal{U}_\varepsilon$ defines a solution to the holonomy Yang–Baxter equation for the Lie group GL_2^*. The braiding on this group is birational and it is given by the factorization mapping.

Moreover, one can show that the category of modules over $\mathcal{U}_\varepsilon$, where objects are finite dimensional modules irreducible over Z_0, is a tensor category fibered over GL_2^*. One can also show that this is true for any factorizable Lie algebra related to a simple Lie algebra: the category of finite dimensional models irreducible over Z_0 is a tensor category fibered over Z_0. These topics will be the subject of a separate publication.

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1. Quantized Universal enveloping algebra of gl_2

1.1. **The algebra $U_h(gl_2)$.**

1.1.1. The algebra $U_h(gl_2)$ over the ring $\mathbb{C}[[h]]$ is generated by elements H, G, X, and Y with the defining relations

$$[H,G]=0, [H,X]=2X, [H,Y]=-2Y, \tag{1}$$

$$[G,X]=2X, [G,Y]=-2Y, \tag{2}$$

$$[X,Y]=\frac{e^{\frac{hH}{2}}-e^{-\frac{hG}{2}}}{e^{\frac{h}{2}}-e^{-\frac{h}{2}}} \tag{3}$$

The Hopf algebra structure on $U_h(gl_2)$ is defined by the action of the comultiplication on the generators:

$$\Delta H = H\otimes 1 + 1\otimes H,\ \Delta G = G\otimes 1 + 1\otimes G, \tag{4}$$

$$\Delta X = X\otimes e^{\frac{hH}{2}}\otimes 1 + 1\otimes X,\ \Delta Y = Y\otimes 1 + e^{-\frac{hG}{2}}\otimes Y \tag{5}$$

Elements H, G, X and Y are deformed versions of $2e_{11},\ 2e_{22},\ e_{12}$ and e_{21} in gl_2.

The algebra $U_h(gl_2)[h^{-1}]$ is isomorphic to the Drinfeld double of the quantized universal enveloping algebra $U_h(b)[h^{-1}]\subset U_h(sl_2)[h^{-1}]$ (see Appendix A). As the double of a Hopf algebra it is quasitriangular with the universal R-matrix [DR] given by the canonical element in $U_h(b)\widehat{\otimes}U_h(b)^\vee$ where $U_h(b)^\vee$ is a dual counterpart of $U_h(b)$ and $\widehat{\otimes}$ is the tensor product completed over formal power series in h:

$$R = \exp(\frac{h}{4}H\otimes G)\prod_{n\geq 0}(1-e^{-\frac{h}{2}}(e^{\frac{h}{2}}-e^{-\frac{h}{2}})^2 X\otimes Ye^{-nh})^{-1} \tag{6}$$

This is the element of $U_h(gl_2)^{\otimes 2}[[h]]$ which one should consider as a formal power series in h.

Element R as the universal R-matrix satisfies the following identities:

$$R\Delta(a)R^{-1} = \sigma\cdot\Delta(a) \tag{7}$$

$$(\Delta\otimes id)(R) = R_{13}R_{23}$$

$$(id\otimes\Delta)(R) = R_{13}R_{12}$$

and, in particular, it satisfies the Yang–Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

1.2. **The inner automorphism $\mathcal{R}$.** Define the inner automorphism $\mathcal{R} : U_h(gl_2)^{\otimes 2}[[h]] \to U_h(gl_2)^{\otimes 2}[[h]]$ as

$$\mathcal{R}(x\otimes y) = R(x\otimes y)R^{-1} \tag{8}$$

It is easy to compute the action of $\mathcal{R}$ on the generators.

Theorem 1. *The following identities hold true:*

$$\mathcal{R}(1\otimes e^{\frac{hH}{2}}) = (1\otimes e^{\frac{hH}{2}})(1-e^{\frac{h}{2}}(e^{\frac{h}{2}}-e^{-\frac{h}{2}})^2 e^{-\frac{hH}{2}}X\otimes Ye^{\frac{hG}{2}})^{-1} \tag{9}$$

$$\mathcal{R}(1\otimes e^{\frac{hG}{2}}) = (1\otimes e^{\frac{hG}{2}})(1-e^{\frac{h}{2}}(e^{\frac{h}{2}}-e^{-\frac{h}{2}})^2 e^{-\frac{hH}{2}}X\otimes Ye^{\frac{hG}{2}})^{-1}$$

$$\mathcal{R}(X\otimes 1) = X\otimes e^{\frac{hG}{2}}$$

$$\mathcal{R}(1\otimes Y) = e^{-\frac{hH}{2}}\otimes Y$$

The proof is immediate by the use of the commutation relations between the generators and the equation

$$f(zq;q) = (1-z)f(z;q)$$

for the function $f(z;q) = (1-z)^{-1}(1-zq)^{-1}(1-zq^2)^{-1}\dots$.

The action of $\mathcal{R}$ on elements $1\otimes X$, $Y\otimes 1$, $e^{\frac{hH}{2}}\otimes 1$, and $e^{\frac{hG}{2}}\otimes 1$ can be derived from the formulae above and from identity (7).

The Yang–Baxter equation for R implies the Yang–Baxter equation for $\mathcal{R}$:

$$\mathcal{R}_{12}\cdot\mathcal{R}_{13}\cdot\mathcal{R}_{23} = \mathcal{R}_{23}\cdot\mathcal{R}_{13}\cdot\mathcal{R}_{12} \tag{10}$$

2. The algebra $\mathcal{U}$

2.1. **The algebra.** The algebra $\mathcal{U}$ is generated over $\mathbb{C}[t,t^{-1}]$ by elements K, L, E and F with the following defining relations

$$KL = LK,\ KE = t^2EK,\ KF = t^{-2}FK,$$

$$LE = t^2EL,\ LF = t^{-2}FL,$$

$$EF - FE = (t-t^{-1})(K-L^{-1})$$

The center of $\mathcal{U}$ is generated freely by Laurent polynomials in KL^{-1} and

$$c = EF + Kt^{-1} + L^{-1}t \tag{11}$$

This is a Hopf algebra with comultiplication

$$\Delta(K) = K\otimes K,\ \Delta(L) = L\otimes L,$$

$$\Delta(E) = E\otimes K + 1\otimes E,\ \Delta(F) = F\otimes 1 + L^{-1}\otimes F\ .$$

The map $\phi : \mathcal{U}\to U_h(gl_2)$ acting on the generators as

$$\phi(K) = \exp(\frac{hH}{2}),\ \phi(L) = \exp(\frac{hG}{2}), \phi(t) = e^{\frac{h}{2}}$$

$$\phi(E) = (e^{\frac{h}{2}} - e^{-\frac{h}{2}})X,\ \phi(F) = (e^{\frac{h}{2}} - e^{-\frac{h}{2}})Y$$

extends to a homomorphism of Hopf algebras.

2.2. **The outer automorphism $\mathcal{R}$.** The algebra $\mathcal{U}$ is not quasitriangular. Instead, there is an outer automorphism of the division ring $\overline{\mathcal{U}^{\otimes 2}}$ of $\mathcal{U}^{\otimes 2}$ which we denote by the same letter $\mathcal{R}$ as the automorphism (8) which acts on the generators as

$$\mathcal{R}(1\otimes K) = (1\otimes K)(1 - tK^{-1}E\otimes FL)^{-1}$$
$$\mathcal{R}(1\otimes L) = (1\otimes L)(1 - tK^{-1}E\otimes FL)^{-1}$$
$$\mathcal{R}(E\otimes 1) = E\otimes L$$
$$\mathcal{R}(1\otimes F) = K^{-1}\otimes F$$

Define its action on the elements $K\otimes 1$, $L\otimes 1$, $K\otimes 1$, and $L\otimes 1$ through the condition that

$$\mathcal{R}(\Delta(a)) = \sigma\circ\Delta(a) \tag{12}$$

where a is any of the generators of $\mathcal{U}$.

It is clear that the homomorphism ϕ brings the outer automorphism $\mathcal{R}$ to (8). It also satisfies the Yang–Baxter relation (10).

3. The algebra $\mathcal{U}_\varepsilon$

3.1. **The algebra.** Let ε be a primitive root of 1 of an odd degree ℓ (this is a technical assumption, convenient because then ε^{2n} runs through all possible ℓ-th roots of 1 for $n = 1,\dots,\ell$). Denote by $\mathcal{U}_\varepsilon$ the specialization of $\mathcal{U}$ to $t=\varepsilon$. The following theorem was proven in [DCK] for any simple Lie algebra.

Theorem 2.
- *Elements E^ℓ, F^ℓ, $K^{\pm\ell}$, and $L^{\pm\ell}$ are central in $\mathcal{U}_\varepsilon$. Denote by Z_0 the central subalgebra in $\mathcal{U}_\varepsilon$ which they generate.*
- *Z_0 is a Hopf subalgebra with the comultiplication*
$$\Delta(K^\ell) = K^\ell\otimes K^\ell,\ \Delta(L^\ell) = L^\ell\otimes L^\ell,$$
$$\Delta(E^\ell) = E^\ell\otimes K^\ell + 1\otimes E^\ell,\ \Delta(F^\ell) = F^\ell\otimes 1 + L^{-\ell}\otimes F^\ell.$$
- *The algebra $\mathcal{U}_\varepsilon$ is a free Z_0-module of dimension ℓ^4.*
- *The center $Z(\mathcal{U}_\varepsilon)$ is generated by Z_0 and by the element (11) modulo the relation*
$$\prod_{j=0}^{\ell-1}(c - K\varepsilon^{j+1} - L^{-1}\varepsilon^{-j-1}) = E^\ell F^\ell$$
- *Let a,b,c,d be coordinates on the group $B_+\times B_-$ such that for $b_\pm\in B_\pm$ we have:*
$$b_+ = \begin{pmatrix} 1 & b \\ 0 & a\end{pmatrix},\ b_- = \begin{pmatrix} d & 0 \\ c & 1\end{pmatrix}$$
Then the map $F^\ell\to b$, $E^\ell\to -cd^{-1}$, $K^\ell\to a$, $L^\ell\to d^2$ is an isomorphism of Hopf algebras $Z_0\to C(B_+\times B_-)$

- *$\mathcal{U}_\varepsilon$ is semisimple over a Zariski open subvariety in $Spec(Z_0) \simeq B_+ \times B_-$.*

3.2. **Z_0-irreducible quotients.** Let $x \in GL_2^*$ be an irreducible Z_0-character and $I_x \subset \mathcal{U}_\varepsilon$ be the corresponding ideal. The quotient algebra

$$A_x = \mathcal{U}_\varepsilon / I_x$$

is finite-dimensional of dimension ℓ^4. There are three natural structures of a left module on A_x. For $a \in \mathcal{U}_\varepsilon$ denote by $[a]$ the class of a in A_x. Then the three actions are:

- $\pi(a)[b] = [ab]$,
- $\phi(a)[b] = [bS(a)]$,
- $\psi(a)[b] = [bS^{-1}(a)]$.

Assume that $x \in GL_2^*$ is generic, i.e. that A_x is semisimple. Fix an isomorphism of algebras $\phi_x : A_x \simeq \oplus_{i=1}^n Mat(k_i)$. For the algebra $\mathcal{U}_\varepsilon$ it is known [DCK, DCKP] that $n = \ell$ and $k_i = \ell$ for all $i = 1, \dots, \ell$. Define

$$t : A_x \to \mathbb{C}, \ t(a) = \sum_{i=1}^{\ell} t_i Tr(\phi_x^i(a))$$

where Tr is the matrix trace in $Mat(k_i)$ and $\phi_x^i : A_x \to Mat(k_i)$ is the i-th component of ϕ_x. It is clear that $t(a)$ does not depend on a particular choice of ϕ_x. Indeed, any other such isomorphism differs from ϕ_x by an inner automorphism of $\oplus_{i=1}^n Mat(k_i)$. Since trace is cyclically invariant, the value of $t(a)$ for such isomorphism will be the same as for ϕ_x. Thus, for generic x we have an invariant bilinear form on A_x:

$$(a, b) = t(ab) \ ,$$

It is non-degenerate if $t_i \neq 0$ for each $i = 1, \dots, n$.

Fix a scalar product on A_x as above. It gives an isomorphism of vector spaces $A_x^* \simeq A_x$. It is easy to verify that the the pairing between $\mathcal{U}_\varepsilon$-modules (A_x, ϕ) and (A_x, π) given by the map $e_x : (A_x, \phi) \otimes (A_x, \pi) \to \mathbb{C}$

$$e_x : a \otimes b \to t(ab) \tag{13}$$

is $\mathcal{U}_\varepsilon$-invariant with respect to the diagonal action. Indeed, by using Sweedler's notation $\Delta(c) = \sum_c c^{(1)} \otimes c^{(2)}$ for the action of the comultiplication on element c, we have:

$$e_x(\sum_c aS(c^{(1)}) \otimes c^{(2)}b) = \sum_c t(aS(c^{(1)})c^{(2)}b) = \varepsilon(c)t(ab)$$

Similarly

$$e_x(\sum_c c^{(1)}a \otimes bS^{-1}(c^{(2)})) = \sum_c t(c^{(1)}abS^{-1}(c^{(2)})) = \varepsilon(c)t(ab)$$

Therefore, the map $e_x : (A_x, \pi) \otimes (A_x, \psi) \to \mathbb{C}$ defined by (13) is also $\mathcal{U}_\varepsilon$-invariant.

Consider the mapping $i_x : \mathbb{C} \to A_x \otimes A_x$ acting at 1 as

$$i_x(1) \mapsto \sum_i e_i \otimes e^i$$

It is easy to see that it is a morphism of $\mathcal{U}_\varepsilon$-modules $\mathbb{C} \to (A_x, \pi) \otimes (A_x, \phi)$. Indeed, let $a_i^j = t(ae_ie^j)$, then

$$\sum_a \sum_i a^{(1)} e_i \otimes e^i S(a^{(2)}) = \sum_a \sum_{i,j,k} (a^{(1)})_i^j \otimes (S(a^{(2)})_k^i e^k = \sum_a \sum_{j,k} (a^{(1)} S(a^{(2)})_k^j e_j \otimes e^k = \varepsilon(a) \sum_i e_i \otimes e^i \tag{14}$$

which implies the first statement. Similarly, the mapping $\mathbb{C} \to (A_x, \psi) \otimes (A_x, \pi)$ is also a morphism of $\mathcal{U}_\varepsilon$-modules.

3.3. **The action of $\mathcal{R}$ on $Z_0 \otimes Z_0$.**

Theorem 3. *The subspace $Z_0 \otimes Z_0 \subset \mathcal{U}_\varepsilon \otimes \mathcal{U}_\varepsilon$ is invariant with respect to the action of the automorphism $\mathcal{R}$.*

Proof. From the action of $\mathcal{R}$ on the generators of $\mathcal{U}_\varepsilon$ and from the relations between generators we have:

$$\mathcal{R}(1 \otimes K^\ell) = (1 \otimes K^\ell)(1 + K^{-\ell} E^\ell \otimes F^\ell L^\ell)^{-1}$$
$$\mathcal{R}(1 \otimes L^\ell) = (1 \otimes L^\ell)(1 + K^{-\ell} E^\ell \otimes F^\ell L^\ell)^{-1}$$
$$\mathcal{R}(E^\ell \otimes 1) = E^\ell \otimes L^\ell$$
$$\mathcal{R}(1 \otimes F^\ell) = K^{-\ell} \otimes F^\ell \ .$$

The comultiplication acts on ℓ-th powers of the generators as

$$\Delta(K^\ell) = K^\ell \otimes K^\ell, \quad \Delta(L^\ell) = L^\ell \otimes L^\ell$$
$$\Delta(E^\ell) = E^\ell \otimes K^\ell + 1 \otimes E^\ell$$
$$\Delta(F^\ell) = F^\ell \otimes 1 + L^{-\ell} \otimes F^\ell \ .$$

These formulae and the defining property $\mathcal{R}(\Delta(a)) = \sigma \circ \Delta(a)$ imply that

$$\mathcal{R}(K^\ell \otimes K^\ell) = K^\ell \otimes K^\ell, \tag{15}$$
$$\mathcal{R}(L^\ell \otimes L^\ell) = L^\ell \otimes L^\ell, \tag{16}$$
$$\mathcal{R}(E^\ell \otimes K^\ell + 1 \otimes E^\ell) = K^\ell \otimes E^\ell + E^\ell \otimes 1, \tag{17}$$
$$\mathcal{R}(E^\ell \otimes 1 + L^{-\ell} \otimes F^\ell) = F^\ell \otimes L^{-\ell} + 1 \otimes F^\ell, \tag{18}$$

These formulas describe implicitly the action of $\mathcal{R}$ on the rest of the generators of $Z_0 \otimes Z_0$. In particular, it is clear that the image is in $Z_0 \otimes Z_0$. ∎

Translating the action of $\mathcal{R}$ on the generators of $Z_0 \otimes Z_0$ through the identification of Z_0 and $C(GL_2^*)$ we have the following statement.

Theorem 4. *The automorphism $\mathcal{R}$ is the pull-back of the birational correspondence $\beta : GL_2^* \times GL_2^* \to GL_2^* \times GL_2^*$ defined as follows:*

$$\beta : (x, y) \mapsto (x_L(x, y), x_R(x, y)) \tag{19}$$

where $I(x_L(x, y)) = x_- I(y) x_-^{-1}$ and $I(x_R(x, y)) = (x_L(x, y))_+^{-1} I(x)(x_L(x, y))_+$. Here $I : GL_2^* \to GL_2$ is the factorization mapping $I(x_+, x_-) = x_+ x_-^{-1}$.

Proof. Let a_1, b_1, c_1, d_1 and a_2, b_2, c_2, d_2 be coordinates of points $x, y \in GL_2^*$ respectively (as in Theorem 2. Let $\alpha_1, \beta_1, \gamma_1, \delta_1$ and $\alpha_2, \beta_2, \gamma_2, \delta_2$ be coordinates of points $x_R, x_L \in GL_2^*$. By definition, the pull-back of the mapping β is $\mathcal{R}$. From the explicit action of $\mathcal{R}$ on the coordinate functions we have

$$\alpha_2 = a_2(1 - c_1 b_1 b_2 d_2 / a_1 d_1 d_2)^{-1}$$

$$\delta_1 = d_1(1 - c_1 b_1 b_2 d_2 / a_1 d_1 d_2)^{-1}$$

$$\gamma_1 \delta_1^{-1} = c_1 d_1^{-1} d_2$$

$$\beta_2 = b_2 a_1^{-1}$$

$$\alpha_1 \alpha_2 = a_1 a_2$$

$$\gamma_1 \delta_1^{-1} \alpha_2 + \gamma_2 \delta_2^{-1} = a_1 c_1 d_1^{-1} + \gamma_2 \delta_2$$

$$\beta_1 + \delta_1^{-1} = b_1 d_2^{-1} + b_2$$

Now, it remains to prove that this mapping can be written as (19). This a simple linear algebra exercise. □

Let I_x be the ideal in $\mathcal{U}_\varepsilon$ corresponding to the irreducible Z_0-character $x \in GL_2^*$. Theorems 3 and 4 have an important implication. The mapping $\mathcal{R}$ induces an isomorphism of algebras

$$\mathcal{R}(x, y) : A_x \otimes A_y \to A_{x_R(x,y)} \otimes A_{x_L(x,y)}$$

This mapping is also an isomorphism of the tensor product of left $\mathcal{U}_\varepsilon$-modules.

4. Braiding for irreducible representations of $\mathcal{U}_\varepsilon$.

4.1. **Restriction of the braiding to irreducible representations.** Let Z_1 be the central subalgebra generated by the central elements c and KL^{-1}. From the definition of $\mathcal{R}$ we have

$$\mathcal{R}(c_1 \otimes c_2) = c_1 \otimes c_2$$

for any $c_{1,2} \in Z_1$. For generic x the algebra A_x is semisimple and its irreducible representations are separated by eigenvalues of c. Let A_x^i be one of the irreducible representations of A_x.

Since $\mathcal{R}$ acts trivially on $Z_1 \otimes Z_1$, the isomorphism $\mathcal{R}(x,y) : A_x \otimes A_y \to A_{x_R(x,y)} \otimes A_{x_L(x,y)}$ restricts to the subalgebra $A_x^i \otimes A_y^j$ and induces an algebra isomorphism

$$\mathcal{R}^{i,j}(x,y) : A_x^i \otimes A_y^j \to A^i_{x_R(x,y)} \otimes A^j_{x_L(x,y)} \ .$$

Algebras A_x^i are fibers of a bundle of algebras with the fiber $Mat(\ell \times \ell)$. This bundle is trivializable over sufficiently small neighborhood of 1. Fix such trivialization, i.e. for each x in this neighborhood fix an algebra isomorphism $\phi_x^i : A_x^i \simeq Mat(\ell)$. Then the mapping $\mathcal{R}^{i,j}(x,y)$ induces an automorphism of $Mat(\ell)^{\otimes 2}$. Since all automorphisms of a matrix algebra are inner, there exists $R^{i,j}(x,y) \in Mat(\ell)^{\otimes 2}$ such that

$$(\phi^i_{x_R(x,y)} \otimes \phi^j_{x_L(x,y)}) \circ \mathcal{R}^{i,j}(x,y) \circ ((\phi_x^i)^{-1} \otimes (\phi_y^i)^{-1})(A \otimes B) = R^{i,j}(x,y)(A\otimes B)R^{i,j}(x,y)^{-1} \ , \tag{20}$$

for any $A, B \in Mat(\ell)$. The Yang–Baxter relation for $\mathcal{R}$ implies the projective holonomy Yang–Baxter equation for $R^{i,j}(x,y)$,

$$R^{i,j}(x'',y'')_{12} R^{i,k}(x,z'')_{13} R^{j,k}(y,z)_{23} = c^{i,j,k}(x,y,z) R^{j,k}(y',z')_{23} R^{i,k}(x',z)_{13} R^{i,j}(x,y)_{12} \tag{21}$$

where x, x', x'', y, y' etc are GL_2^*-colorings of the diagrams of Fig. 1 (see [KR1, KR2]). In terms of the mapping $\beta : (x,y) \mapsto (x_L(x,y), x_R(x,y)$ the arguments in this equation are given by

$$z'' = x_L(y,z),\ y'' = x_R(y,z),\ x'' = x_R(x,z''),\ x' = x_R(x,y),\ y' = x_L(x,y),\ z' = x_L(x',z).$$

The function $c^{i,j,k}(x,y,z)$ can be determined from the comparison of determinants of the left and right hand sides of the equation. We will describe a normalization of $R^{i,j}(x,y)$ for which $c^{i,j,k}(x,y,z) = 1$.

4.2. **Irreducible representations of $\mathcal{U}_\varepsilon$.** Now we fix the isomorphisms $\phi_x^i : A_x^i \to Mat(\ell)$ by choosing a specific realization of irreducible representations of $\mathcal{U}_\varepsilon$. This realization is known as a cyclic representation.

Let u, v and x be variables such that

$$K^\ell = u^\ell v^\ell,\ L^\ell = u^{-\ell} v^\ell,\ c = u(x + x^{-1}),\ E^\ell = y^\ell \tag{22}$$

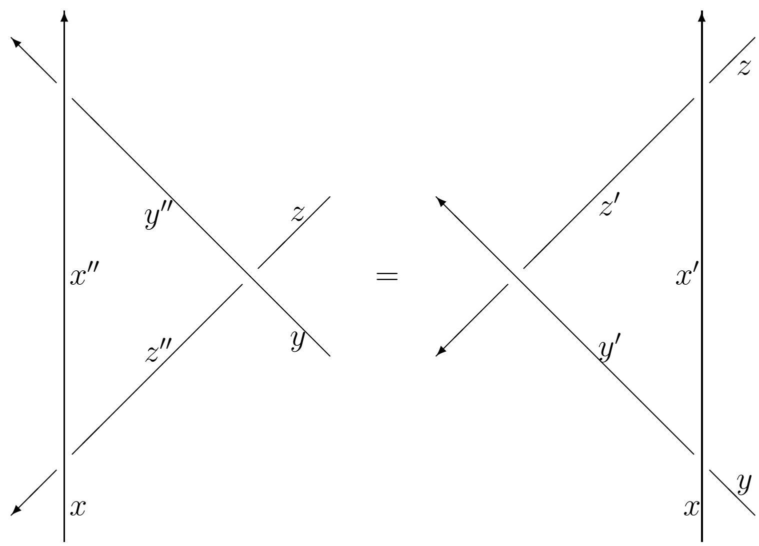

FIGURE 1. Holonomy Yang–Baxter equation.

The following representation of $\mathcal{U}_\varepsilon$ in $\mathbb{C}^\ell$ with the linear basis $\{v_m\}_{m=1}^{\ell}$ is irreducible:

$$Kv_m = uv\varepsilon^{2m}v_m,\ Lv_m = u^{-1}v\varepsilon^{2m}v_m,\ Ev_m = yv_{m+1} \tag{23}$$

$$Fv_m = y^{-1}u(xv^{-1}\varepsilon^{-2m+1}-1)(v\varepsilon^{2m-1}-x^{-1})v_{m-1} \tag{24}$$

Indeed, since K and L are diagonal in this representation, they are diagonal in any subrepresentation. Taking into account the fact that E and F act as cyclic shift operators in the eigen-basis of K and L, one deduces that the only possible subrepresentaions are the trivial one and the representation itself. It is easy to see that in this representation

$$F^\ell = u^\ell(x^\ell v^{-\ell}-1)(v^\ell - x^{-\ell})$$

The parameters (22) of the irreducible representation are related to the coordinates a, b, c, d on GL_2^* by the following formulae

$$a = u^\ell v^\ell, d = u^{-\ell}v^\ell, c = -y^\ell u^{-\ell}v^\ell, b = y^{-\ell}u^\ell(x^\ell - v^\ell - v^{-\ell} + x^{-\ell})$$

Remark 1. *We have the identities*

$$Tr(b_+(b_-)^{-1}) = a + d^{-1} - bcd^{-1} = u^\ell(x^\ell - x^{-\ell})$$

and

$$det(b_+(b_-)^{-1}) = ad^{-1} = u^{2\ell}$$

Proposition 1. (1) *Representations with different branches of $v \to v^\ell$ and $y \to y^\ell$ are isomorphic.*

(2) *Branches of* $u \to u^\ell$, $x \to x^\ell$ *parameterize isomorphism classes of irreducible representations for generic values of* a, b, c, d.

The proof is straightforward.

Let A and B be $\ell \times \ell$ matrices acting on the standard basis in $\mathbb{C}^\ell$ as follows:

$$Av_n = \varepsilon^{2n} v_n, \ Bv_n = v_{n+1}$$

which satisfy the relations

$$A^\ell = 1, \ B^\ell = 1, \ AB = \varepsilon^2 BA,$$

Then the representation (23) can be written as the following homomorphism of algebras

$$K \mapsto uvA, L \mapsto u^{-1}vA, E \mapsto yB, \tag{25}$$

$$F \mapsto y^{-1}u(xv^{-1}\varepsilon A^{-1} - 1)(vA\varepsilon^{-1} - x^{-1})B^{-1} \tag{26}$$

The collection of these homomorphisms is a trivialization $\phi^i_x : A^i_x \to Math(\ell)$ over a sufficiently small neighborhood of 1 in GL_2^*.

4.3. **The braiding for generic irreducible modules.** The action of $\mathcal{R}$ on the tensor product of two irreducible representations is the specialization of the formulae from section 2.2. Substituting this action into (20) we arrive to the following equations for R

$$\begin{aligned} R(1 \otimes A)R^{-1} &= u_2v_2\tilde{u}_2^{-1}\tilde{v}_2^{-1}(1 \otimes A) \\ &\times (1 - \varepsilon u_1^{-1}v_1^{-1}y_1y_2^{-1}\tilde{v}_2^{-1}A^{-1}B \otimes (x_2v_2^{-1}\varepsilon A^{-1} - 1)(v_2A\varepsilon^{-1} - x_2)B^{-1}A))^{-1}, \end{aligned} \tag{27}$$

$$\begin{aligned} \tilde{y}_2^{-1}\tilde{u}_2R(1 \otimes (\tilde{x}_2\tilde{v}_2^{-1}\varepsilon A^{-1} - 1)(\tilde{v}_2A\varepsilon^{-1} - \tilde{x}_2)B^{-1})R^{-1} & \\ = u_1^{-1}v_1^{-1}(A^{-1} \otimes y_2^{-1}u_2(x_2v_2^{-1}\varepsilon A^{-1} - 1)(v_2A\varepsilon^{-1} - x_2)B^{-1}), & \end{aligned} \tag{28}$$

$$\tilde{u}_1\tilde{v}_1\tilde{u}_2\tilde{v}_2R(A \otimes A)R^{-1} = u_1v_1u_2v_2(A \otimes A), \tag{29}$$

$$\tilde{y}_1R(B \otimes 1)R^{-1} = y_1u_2v_2(B \otimes A), \tag{30}$$

Here $\tilde{u}_1^2 = u_1^2$ and $\tilde{u}_2^2 = u_2^2$.

For sufficiently small x, v, and z such that $z^\ell = (x^\ell v^{-\ell} - 1)(v^\ell - x^\ell)$ define a linear operator $U : \mathbb{C}^\ell \to \mathbb{C}^\ell$

$$Uv_n = z \prod_{m=1}^{n} ((xv^{-1}\varepsilon^{-2m+1} - 1)^{-1}(v\varepsilon^{2m-1} - x^{-1})^{-1}v_n$$

Let P_n be the projector on the vector v_n in $\mathbb{C}^\ell$:

$$P_nv_m = \delta_{n,m}v_m,$$

Define an operator D acting in $\mathbb{C}^\ell \otimes \mathbb{C}^\ell$ as

$$D(v_n \otimes v_m) = \varepsilon^{2nm} \chi_1^{-n} \chi_2^m (v_n \otimes v_m)$$

Assume that the parameters $\chi_1, \chi_2, \varepsilon^a$ satisfy the following relations

$$u_1 v_1 u_2 v_2 = \chi_1 \varepsilon^{-2a} \tilde{u}_1 \tilde{v}_1 \tilde{u}_2 \tilde{v}_2$$

$$y_1 u_2 v_2 \chi_2 = \tilde{y}_1$$

$$u_1^{-1} v_1^{-1} z_2 \chi_2 \varepsilon^{-2a} = \tilde{z}_2 \tilde{y}_2^{-1} \tilde{u}_2$$

and define a power series in z given by the expansion of the function

$$\Phi(z) = \prod_{m=1}^{\ell} (1 - \varepsilon^{2m} z)^{-\frac{m}{\ell}} \tag{31}$$

Theorem 5. *For generic $x, y \in GL_2^*$ any solution to the equations (27)—(30) is a scalar multiple of*

$$R = D(B^a \otimes U) R_1 (1 \otimes \tilde{U}^{-1}) \tag{32}$$

where $R_1 = \Phi(sB \otimes B^{-1}\varepsilon^{-2})$, $s = u_2 v_2 \tilde{u}_2^{-1} \tilde{v}_2^{-1}$, D is defined above and U and $\tilde{U}$ depend on parameters z_2, u_2, v_2, and $\tilde{z}_2, \tilde{u}_2, \tilde{u}_1$, respectively.

Proof. The operator U satisfies the identity

$$U^{-1}(xv^{-1}\varepsilon A^{-1} - 1)(vA\varepsilon^{-1} - x)B^{-1}U = B^{-1}$$

Write R as

$$R = D(B^a \otimes U) R_1 (1 \otimes \tilde{U}^{-1})$$

Here operator $\tilde{U}$ depends on $\tilde{z}_2, \tilde{u}_2, \tilde{v}_2$ and U depends on z_2, u_2, v_2. Then, the equations for R imply the following relations for the operator R_1:

$$R_1 (A \otimes A) R_1^{-1} = A \otimes A \ , \tag{33}$$

$$R_1 (1 \otimes A) R_1^{-1} = t(1 \otimes A)(1 - sB \otimes B)^{-1} \tag{34}$$

$$R_1 (1 \otimes B^{-1}) R_1^{-1} = 1 \otimes B^{-1} \tag{35}$$

$$R_1 (B \otimes 1) R_1^{-1} = B \otimes 1, \tag{36}$$

where

$$s = u_2 v_2 \tilde{u}_2^{-1} \tilde{v}_2^{-1}, \ t = \varepsilon \chi_2 \chi_3 u_1^{-1} v_1^{-1} y_1 y_2^{-1} \tilde{v}_2^{-1}$$

Identities (15)—(18) imply that $t^\ell = 1 - s^\ell$.

The function Φ satisfies the following "difference equation":

$$\Phi(\varepsilon^2 z) = (1 - z^\ell)^{\frac{1}{\ell}} (1 - \varepsilon^2 z^{-1})^{-1} \Phi(z) \tag{37}$$

From equations (33), (35), (36) we conclude that R_1 is a polynomial in $B \otimes B^{-1}$. Equation (34) determines the coefficients of this polynomial up to an overall scalar factor. Taking into account the difference equation for ϕ, we conclude that there is one dimensional family of solutions to linear equations (34):

$$R_1 = const\ \Phi(sB \otimes B^{-1}\varepsilon^{-2})$$

□

Remark 2. *The power series $\Phi(z)$ can be analytically continued to the meromorphic function $\Phi(z)$ defined in a neighborhood of $z = 0$. More precisely Φ determines the curve $C_\Phi = \{\Phi^\ell \prod_{m=1}^{\ell}(1-\varepsilon^{2m}z)^{-\frac{m}{\ell}} = 1\}$ in $\mathbb{C}\times\mathbb{C}$. The difference equation is the action of $\mathbb{Z}_\ell$ on the curve $C_1 = \{(\Phi,\varepsilon,\phi) \in \mathbb{C}^3 | \Phi^\ell \prod_{m=1}^{\ell}(1-\varepsilon^{2m}z)^{-\frac{m}{\ell}} = 1, \phi^\ell + z^\ell = 1\}$. The element $\omega \in \mathbb{Z}_\ell$ acts as $(\Phi,\varepsilon,\phi) \to (\Phi\phi(1-\omega z)^{-\ell}, z\omega, \phi)$.*

It is not difficult to compute the determinant of the matrix R:

$$det(R) = (1-s^\ell)^{\frac{\ell(\ell+2)}{2}}\ ,$$

Since R is fixed yet only up a scalar function of parameters of the representations, then the element

$$\tilde{R} = (1-s^\ell)^{\frac{\ell+1}{2\ell}} R$$

intertwines the same representations as R with the property $det(\tilde{R}) = 1$ and at $s = 0$ the matrix $\tilde{R} = D_{12}B_1^a U_2 \tilde{U}_2^{-1}$ satisfies the Yang–Baxter equation. Therefore the factor c^{ijk} is trivial for the matrix $\tilde{R}$.

Thus, every pair of generic irreducible $\mathcal{U}_\varepsilon$-modules defines the following solution to the holonomy Yang–Baxter equation:

$$R^{i,j}(x,y) = (1-s^\ell)^{\frac{\ell+1}{2\ell}} D(B^a \otimes U)\Phi(sB \otimes B^{-1}\varepsilon^2)(1 \otimes \tilde{U}^{-1})\ .$$

Here x and y are in sufficiently small neighborhood of 1 in GL_2^* and all other ingredients are defined above.

Appendix A

1. Let $b \subset sl_2$ be the Borel subalgebra and H, the generator of the Cartan subalgebra. The algebra $U_h b$ is generated by H and X with the defining relation

$$HX - XH = 2X.$$

This is a Hopf algebra over $\mathbb{C}[[h]]$ with the comultiplication

$$\begin{aligned} \Delta H &= H \otimes 1 + 1 \otimes H, \\ \Delta X &= X \otimes e^{\frac{hH}{2}} + 1 \otimes X. \end{aligned}$$

2. Let $(U_hb)^0$ be the Hopf algebra generated by elements $H^\vee,\ X^\vee$, completed by formal power series in h and $H^\vee$ with the defining relations

$$\begin{aligned} [H^\vee, X^\vee] &= -\frac{h}{2}X^\vee, \\ \Delta H^\vee &= H^\vee \otimes 1 + 1 \otimes H^\vee, \\ \Delta X^\vee &= X^\vee \otimes 1 + e^{-2H^\vee} \otimes X^\vee, \end{aligned}$$

3. There is a nondegenerate pairing $\langle \cdot, \cdot \rangle \colon U_hb \otimes (U_hb)^0 \to \mathbb{C}[[h]]$ of Hopf algebras with

$$\langle l, ba\rangle = \langle \Delta l, a \otimes b\rangle \tag{38}$$

$$\langle lm, a\rangle = \langle l \otimes m, \Delta a\rangle \tag{39}$$

This pairing is defined on generators as

$$\begin{aligned} \langle H^\vee, H\rangle &= 1,\ \langle X^\vee, X\rangle = 1 \\ \langle H^\vee, X\rangle &= \langle X^\vee, H\rangle = 0. \end{aligned}$$

It is easy to extend this pairing to the whole algebra using (38). The pairing between the monomials is

$$\langle (H^\vee)^n (X^\vee)^m, H^{n'}, X^{m'}\rangle = \delta_{nn'}\delta_{mm'} n! b_m$$

where

$$b_m = \frac{1-e^{-hn}}{1-e^{-h}} \cdot \frac{1-e^{-h(n-1)}}{1-e^{-h}} \cdots \frac{1-e^{-h}}{1-e^{-h}}.$$

Indeed, it is clear that the right-hand side is proportional to $\delta_{n,n'}\delta_{mm'}$. Let us commute the coefficient. Using the Hopf properties (38) of the pairing and the grading arguments we obtain:

$$\begin{aligned} \langle (H^\vee)^n (X^\vee)^m, H^n X^m\rangle &= \langle (H^\vee)^n \otimes (X^\vee)^m, (\Delta H)^n (\Delta X)^m\rangle \\ &= \langle (H^\vee)^n \otimes (X^\vee)^m, H^n \otimes X^m\rangle \\ &= \langle (H^\vee)^n, H^n\rangle \langle (X^\vee)^m, X^m\rangle. \end{aligned}$$

For the first factor we have

$$\begin{aligned} \langle (H^\vee)^n, H^n\rangle &= \langle (H^\vee)^{n-1} \otimes H^\vee, (\Delta H)^n\rangle \\ &= \langle (H^\vee)^{n-1} \otimes H^\vee, nH^{n-1} \otimes H\rangle \\ &= n\langle (H^\vee)^{n-1}, H^n\rangle \end{aligned}$$

therefore

$$\langle (H^\vee)^n, H^n\rangle = n!$$

For the second factor we have

$$\begin{aligned}\langle (X^\vee)^n, X^n\rangle &= \langle (X^\vee)^{n-1}\otimes X^\vee, (\Delta X)^n\rangle \\ &= \langle (X^\vee)^{n-1}\otimes X^\vee, \frac{-e^{-hn}}{1-e^{-h}}X^{n-1}\otimes e^{\frac{hH}{2}(n-1)}X\rangle.\end{aligned}$$

Here we used the second term in

$$\begin{aligned}(\Delta X)^n &= (X\otimes e^{\frac{hH}{2}}+1\otimes X)^n \\ &= X^n\otimes e^{\frac{hnH}{2}} \\ &+ (X^{n-1}\otimes e^{\frac{h(n-1)H}{2}}X + X^{n-1}\otimes e^{\frac{hH}{2}}Xe^{\frac{h(n-2)}{2}H}+\cdots+X^{n-1}\otimes Xe^{\frac{h(n-1)H}{2}})+\dots \\ &= X^n\otimes e^{\frac{hnH}{2}}+(1+e^{-h}+\cdots+e^{-(n-1)h})X^{n-1}\otimes e^{\frac{(n-1)}{2}hH}X+\dots\end{aligned}$$

Therefore

$$\langle (X^\vee)^n, X^n\rangle = \frac{1-e^{-hn}}{1-e^{-h}}\langle (X^\vee)^{n-1}, X^{n-1}\rangle\cdot\langle X^\vee, e^{\frac{hH(n-1)}{2}}X\rangle.$$

Taking into account that

$$\langle X^\vee, e^{\frac{hH}{2}(n-1)}X\rangle = \langle \Delta X^\vee, X\otimes e^{\frac{hH}{2}(n-1)}\rangle = 1$$

we obtain

$$\langle (X^\vee)^n, X^n\rangle = b_n.$$

where b_n is defined above.

4. The double of the Hopf algebra U_hb is the Hopf algebra structure on the space $\mathcal{D}(U_hb)=U_hb\otimes(U_hb)^0$ completed in formal power series in h and $H^\vee$ such that the coalgebra structure is the tensor product of coalgebras. The algebra structure is completely determined by the condition that the natural embeddings $U_hb,(U_hb)^0\hookrightarrow\mathcal{D}(U_hb)$ are Hopf algebra homomorphisms and that the canonical element $R=\sum_i e_i\otimes e^i$ intertwines the coproduct with the opposite coproduct

$$R\Delta(a)=\Delta^{op}(a)R,$$

where $\Delta^{op}(a)=\sigma_0\Delta(a)$, $\sigma(x\otimes y)=y\otimes x$.

Equivalently, the product in $\mathcal{D}(U_hb)$ can be defined as

$$(a\otimes l)(b\otimes m)=\sum ab_{(2)}\otimes l_{(2)}m<b_{(1)},S^{-1}_{A^*}(l_{(1)})><b_{(3)},l_{(3)}> \tag{40}$$

Here we use the notation $\Delta(a)=\sum a_{(1)}\otimes a_{(2)}$ for the comultiplication in U_hb and $\Delta^o(l)=\sum l_{(2)}\otimes l_{(1)}$ for the comultiplication in $(U_hb)^o$ (see [Ma]).

Using either of these definitions, it is easy to show that $\mathcal{D}(U_hb)$ is isomorphic to the algebra generated by $H,X,H^\vee,X^\vee$, completed in formal power series in h and $H^\vee$ with the defining

relations

$$\begin{aligned}
[H, H^\vee] &= 0,\ [H, X] = 2X,\ [H^\vee, X] = \frac{h}{2}X \\
[H, X^\vee] &= -2X^\vee,\ [H^\vee, X^\vee] = -\frac{h}{2}X^\vee \\
[X, X^\vee] &= e^{\frac{hH}{2}} - e^{-2H^\vee}.
\end{aligned}$$

The universal R-matrix is the canonical element $R = \sum_i e_i \otimes e^i \in U_h b \otimes (U_h b)^0$

$$R = \sum_{n,m\geq 0} \frac{1}{n! b_m} H^n X^m \otimes (H^\vee)^n (X^\vee)^m \tag{41}$$

$$= \exp(H \otimes H^\vee) f(X \otimes X^\vee; e^{-h}) \tag{42}$$

where

$$f(z;q) = \sum_{n\geq 0} \frac{(1-q)^n z^n}{(1-q)\ldots(1-q^n)} = \prod_{n\geq 0}(1-(1-q)zq^n)^{-1}.$$

Here b_m is as above and $q = e^{-h}$. It follows immediately from the definition of the double that

$$\begin{aligned}
(\Delta \otimes id)(R) &= R_{13}R_{23} \\
(id \otimes \Delta)(R) &= R_{13}R_{12}
\end{aligned}$$

and, in particular, that R satisfies the Yang–Baxter equation.

5. The quantum gl_2, or $U_h gl_2$ is the Hopf algebra generated by H, G, X, Y with the defining relations and the coproduct described in (1)–(3). It is clear that the mapping $\varphi : \mathcal{D}(U_h b) \to U_h gl_2$ acting on generators as

$$\varphi(H) = H,\ \varphi(H^\vee) = \frac{h}{4}G,\ \varphi(X) = X,\ \varphi(X^\vee) = Y(e^{\frac{h}{2}} - e^{-\frac{h}{2}})$$

extends to a Hopf algebra homomorphism. The element (6) is the image of the universal R-matrix (41) and therefore satisfies (7).

Appendix B

In section 2.2 we defined the action of $\mathcal{R}$ on the generators $E \otimes 1$, $1 \otimes F$, $1 \otimes K$, and $1 \otimes L$ explicitly and on other generators implicitly by requiring the intertwining property (12). The action of $\mathcal{R}$ on element $F \otimes 1$ and on $1 \otimes E$ can also be found from the condition that $\mathcal{R}(c_1 \otimes c_2) = c_1 \otimes c_2$ for central elements c_1 and c_2 and $\mathcal{R}(K \otimes K) = K \otimes K$. Let us verify this.

First, notice that $\mathcal{R}(K \otimes K) = K \otimes K$ together with (9) implies that

$$\mathcal{R}(K \otimes 1) = (1 - tK^{-1}E \otimes FL)(K \otimes 1)$$

Now let us find the action of $\mathcal{R}$ on $1\otimes E$:

$$
\begin{aligned}
(43)\quad \mathcal{R}(1\otimes E) = \mathcal{R}(1\otimes EF)(K\otimes F^{-1}) = \\
\mathcal{R}(1\otimes (c-t^{-1}K-tL^{-1}))(K\otimes F^{-1}) = \\
1\otimes cF^{-1} - (t^{-1}\mathcal{R}(1\otimes K) + t\mathcal{R}(1\otimes L^{-1}))(K\otimes F^{-1}) = \\
1\otimes E + t^{-1}K\otimes KF^{-1} + tK\otimes L^{-1}F^{-1} - \\
t^{-1}(1\otimes K)(1-tK^{-1}E\otimes FL)^{-1}(K\otimes F^{-1} - t(1-tK^{-1}E\otimes FL)(1\otimes L^{-1}F^{-1}) \\
= 1\otimes E + E\otimes K - (1-t^{-1}K^{-1}E\otimes FL)^{-1}E\otimes KL,
\end{aligned}
$$

Similarly, we have:

$$
\begin{aligned}
(44)\quad \mathcal{R}(F\otimes 1) = (E^{-1}\otimes L^{-1})\mathcal{R}(EF\otimes 1) = \\
(E^{-1}\otimes L^{-1})\mathcal{R}((c-t^{-1}K-tL^{-1})\otimes 1) = \\
E^{-1}c\otimes L^{-1} - t^{-1}(E^{-1}\otimes L^{-1})(\mathcal{R}(K\otimes 1) + t\mathcal{R}(L^{-1}\otimes 1)) = \\
F\otimes L^{-1} + t^{-1}E^{-1}K\otimes L^{-1} + tE^{-1}L^{-1}\otimes L^{-1} - \\
t^{-1}(E^{-1}\otimes L^{-1})(1-tK^{-1}E\otimes FL)(K\otimes 1) - t(E^{-1}L^{-1}\otimes L^{-1})(1-tK^{-1}E\otimes FL) \\
= F\otimes L^{-1} + 1\otimes F - (KL^{-1}\otimes F)(1-tK^{-1}E\otimes FL)^{-1}.
\end{aligned}
$$

Taking into account the action of $\mathcal{R}$ on $K\otimes F$ and $E\otimes L^{-1}$, it is easy to see that the previous formulae are equivalent to

$$\mathcal{R}(E\otimes K + 1\otimes E) = K\otimes E + E\otimes 1,$$

$$\mathcal{R}(F\otimes 1 + L^{-1}\otimes F) = 1\otimes F + F\otimes L^{-1}$$

References

[DCK] C. De Concini, and V. Kac, Representations of quantum groups at roots of 1. Modern quantum field theory (Bombay, 1990), 333–335, World Sci. Publishing, River Edge, NJ, 1991.

[DCKP] C. De Concini, V. Kac, and C. Pocesi, Quantum coadjoint action. J. Amer. Math. Soc. 5 (1992), no. 1, 151–189.

[DR] V. Drinfeld, Quantum groups. Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), 798–820, Amer. Math. Soc., Providence, RI, 1987.

[R] N. Reshetikhin, Quasitriangularity of quantum groups at roots of 1. Comm. Math. Phys. 170 (1995), no. 1, 79–99.

[Ga] F. Gavarini, Geometrical meaning of R-matrix action for quantum groups at roots of 1. Comm. Math. Phys. 184 (1997), no. 1, 95–117.

[T1] V. Turaev,Quantum invariants of knots and 3-manifolds. de Gruyter Studies in Mathematics, 18. Walter de Gruyter and Co., Berlin, 1994.

[T2] V. Turaev, Homotopy field theory in dimension 3 and crossed group-categories. math.GT/0005291

[KR1] R. Kashaev, and N. Reshetikhin, Invariants of tangles with flat connections in their complements.I. Invariants and holonomy R-matrices.math.AT/0202211.

[KR2] R. Kashaev, and N. Reshetikhin, Invariants of tangles with flat connections in their complements. Proceedings Symposia of Pure mathematics, "Graphs and Patters in Mathematics and Theoretical Physics", (M. Lyubich and L. Takhtajan, eds.), AMS, 2004.

[Ma] S. Majid, Foundations of quantum group theory. Cambridge University Press, Cambridge, 1995.

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